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Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A global positive solution of a delay differential equation with indefinite coefficients

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ARTICLE INFO

Article history:

Received 24 August 2011

Accepted 12 March 2012

Keywords:

General theory

Linear functional differential equations

ABSTRACT

In this note, we show the existence of a global positive solution for a class of first-order delay differential equations:

$$x'(t) + a_1(t)x(t) + a_2(t)x(t - h(t)) = 0.$$

In this study, we allow a_1 and a_2 to be of oscillating nature.

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1. Introduction and a theorem

In [1], the authors studied the following delay logarithmic equation:

$$y'(t) = y(t)[F(t) - a_1(t) \log y(t) - a_2(t) \log y(t - h(t))]. \quad (1.1)$$

Eq. (1.1) was proposed by Gopalsamy [2] for describing the model of a single population, where y is the size of the population, $F(t)$ is the growth rate when there are several resources and there is no intra-species competition for these resources, $a_1(t)$ is the measurement of the competition among the individuals, $a_2(t)$ is added to generalize the model to yield competitive effects and $h(t)$ is a maturation delay in the sense that competition involves adults who have matured by an age of $h(t)$ units. One can see that under the invariant transformation $y(t) = e^{x(t)}$, (1.1) reduces to

$$x'(t) + a_1(t)x(t) + a_2(t)x(t - h(t)) = F(t). \quad (1.2)$$

The purpose of this note is to show the existence of a global positive solution of (1.2) for $t \geq t_0 > 0$, when $F(t) \equiv 0$, i.e.,

$$x'(t) + a_1(t)x(t) + a_2(t)x(t - h(t)) = 0, \quad t \geq t_0 > 0. \quad (1.3)$$

Let h be a bounded, positive function on (t_0, ∞) such that $h'(t)$ exists and is bounded on (t_0, ∞) . Let us set $T_0 = \inf_{t \geq t_0} (t - h(t))$.

We associate (1.3) with the initial condition

$$x(t) = \xi(t), \quad \forall t \in [T_0, t_0], \quad (1.4)$$

where ξ is any given initial function. Let $\xi \in C([T_0, t_0], \mathbb{R})$. By a solution of (1.3), we mean a function which is defined on $[T_0, \infty)$, continuous on $[T_0, t_0]$, satisfies $x(t) = \xi(t) \forall t \in [T_0, t_0]$, is continuously differentiable on $[t_0, \infty)$ and satisfies (1.3). We refer the reader to [3] for the existence of a positive solution of (1.3), where $a_1(t) \equiv 0$ and $a_2(t) \geq 0$.

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We make the following hypotheses for a_1 and a_2 :

(H1) Suppose $a'_1(t)$ exists and is a bounded function on $[T_0, \infty)$ with $a_1(t) \rightarrow 0$ as $t \rightarrow \infty$.

(H2) Suppose $a'_2(t)$ exists and is bounded on (t_0, ∞) , and that a_2 is continuous on $[T_0, \infty)$ with $a_2(t) \rightarrow 0$ as $t \rightarrow \infty$.

In this note, we establish the existence of a global positive solution to (1.3) by Schauder's fixed point theorem.

Schauder's fixed point theorem: Let X be a Banach space and U be any nonempty closed, convex subset of X . If M is a continuous mapping of U into itself and MU is relatively compact, then the mapping M has at least one fixed point.

Let $BC([T_0, \infty), \mathbb{R})$ be the Banach space of all bounded continuous real-valued functions on the interval $[T_0, \infty)$ endowed with the supremum norm defined by

$$\|v\| = \sup_{t \geq T_0} |v(t)| \quad \text{for } v \in BC([T_0, \infty), \mathbb{R}).$$

A set S of real-valued functions defined on the interval $[T_0, \infty)$ is called equiconvergent at ∞ if all functions in S are convergent in \mathbb{R} at the point ∞ and for every $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that for all functions y in S ,

$$|y(t) - \lim_{s \rightarrow \infty} y(s)| < \epsilon \quad \text{for every } t \geq T.$$

The following compactness criterion can be seen as an adaptation of a lemma due to Avramescu [4].

The compactness criterion: Let F be an equicontinuous and uniformly bounded subset of the Banach space $BC([T_0, \infty), \mathbb{R})$. If F is equiconvergent at ∞ , then F is relatively compact.

In view of the above compactness criterion, using the classical Schauder theorem, Philos et al. [5] established the existence of least one global solution for second-order nonlinear delay differential equations, which is asymptotic to any given line. We also use the same criterion and prove the existence of a global positive solution of (1.3) in the following.

Theorem 1.1. Let (H1) and (H2) hold. Then (1.3) has a global positive solution x such that

$$x(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. Suppose $C_b([T_0, \infty))$ denotes a Banach space of all bounded continuous real-valued functions on the interval $[T_0, \infty)$. Let us consider

$$A: C_b([T_0, \infty)) \rightarrow C_b([T_0, \infty))$$

defined by

$$(A\phi)(t) = \begin{cases} a_1(t) + a_2(t)e^{\int_{t-h(t)}^t \phi(s)ds}, & t \in (t_0, \infty), \\ (A\phi)(t_0), & t \in [T_0, t_0]. \end{cases}$$

Then A is well-defined. Since a_1 and a_2 are bounded continuous functions on $[T_0, \infty)$ with $a_1(t) \rightarrow 0$, $a_2(t) \rightarrow 0$ as $t \rightarrow \infty$, then there exist $M > 0$, $M_1 > 0$ such that $|a_1(t)| \leq M$, $|a_2(t)| \leq M_1$, $\forall t \geq T_0$. Also, h is bounded, so there exists some $M_2 > 0$ such that $|h(t)| \leq M_2$, $\forall t \geq T_0$. Let us choose $\alpha > 0$ such that $M + M_1 e^{M_2 \alpha} \leq \alpha$ (indeed, one can choose α easily, when M and M_1 are sufficiently small) and define

$$U = \{\psi \in C([T_0, \infty)) \mid |\psi(t)| \leq \alpha\}.$$

It is easy to see that U is a closed, convex set in $C([T_0, \infty))$ and U is invariant under A , i.e.,

$$AU \subset U.$$

By an application of Schauder's fixed point theorem, we aim to prove the existence of a fixed point for A in U . Now we claim that $A(U)$ is relatively compact in U , i.e., if $\{u_n\}$ is any arbitrary sequence in U , then $\{Au_n\}_{n \geq 1}$ has a convergent subsequence in U . For, let $t \in (t_0, \infty)$, then

$$\left| \frac{d}{dt} (Au_n)(t) \right| = \left| a'_1(t) + a_2(t)e^{\int_{t-h(t)}^t u_n(s)ds} [u_n(t) - u_n(t-h(t))(1-h'(t))] + a'_2(t)e^{\int_{t-h(t)}^t u_n(s)ds} \right| \leq C, \quad n \geq 1, \text{ for some } C > 0.$$

When $t \in [T_0, t_0]$, then $\frac{d}{dt} (Au_n)(t) = 0$. An application of the mean-value theorem implies that

$$|(Au_n)(s) - (Au_n)(t)| \leq M|s - t|, \quad \forall s, t > T_0, n \geq 1 \quad (1.5)$$

and therefore the family $\{Au_n\}_{n \geq 1}$ is equicontinuous. Since $AU \subset U$, then $\{Au_n\}_{n \geq 1}$ is uniformly bounded in U . Also

$$\begin{aligned} \lim_{s \rightarrow \infty} Au_n(s) &= \lim_{s \rightarrow \infty} [a_1(s) + a_2(s)e^{\int_{s-h(s)}^s u_n(y)dy}] \\ &= 0 + \lim_{s \rightarrow \infty} a_2(s) \lim_{s \rightarrow \infty} e^{\int_{s-h(s)}^s u_n(y)dy} = 0, \end{aligned}$$

and hence $\{Au_n\}_{n \geq 1}$ is equiconvergent at ∞ . By an application of compactness criteria (the Arzela–Ascoli theorem), $\{Au_n\}_{n \geq 1}$ has a convergent subsequence in U . By following similar lines to the proof of equicontinuity of $\{Au_n\}$, one can show easily that $A: U \rightarrow U$ is continuous and hence by the classical Schauder fixed point theorem, there exists $\phi \in U$ such that $A\phi = \phi$, i.e.,

$$\phi(t) = a_1(t) + a_2(t)e^{\int_{t-h(t)}^t \phi(s)ds}, \quad t \geq t_0.$$

With the setting

$$x(t) = e^{-\int_{t_0}^t \phi(s)ds}, \quad t \geq T_0,$$

one can see that x is a global positive solution of (1.3) with

$$x(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This completes the proof of this theorem. \square

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